Grade 7/8 Math Circles

March 20/21/22/23, 2023

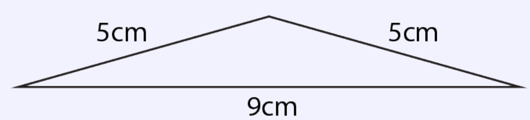
Trigonometry - Solutions

Exercise Solutions

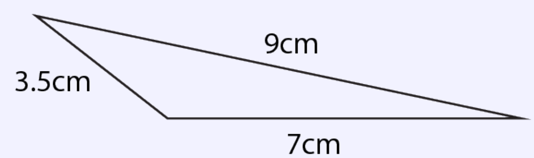
Exercise 1

Label the following triangles using the above terms. More than one can be applied!

a) Triangle A



b) Triangle B



Exercise 1 Solution

a) Triangle A is an obtuse, isosceles triangle.

b) Triangle B is an obtuse, scalene triangle.

Exercise 2

Determine which of the following triangles can be a right triangle:

a) Triangle A with side lengths of 3cm, 5cm and 6cm

b) Triangle B with side lengths of 12ft, 9ft and 15ft

Exercise 2 Solution

a) This is *not* a right triangle since it does not satisfy the Pythagorean Theorem:

$$\begin{aligned} a^2 + b^2 &= 3^2 + 5^2 \\ &= 9 + 25 \\ &= 34 \end{aligned}$$

$$\text{but } c^2 = 6^2 = 36 \neq 34$$

b) This is indeed a right triangle:

$a^2 + b^2 = 9^2 + 12^2$
$a^2 + b^2 = 81 + 144$
$a^2 + b^2 = 225$

$c^2 = 15^2$
$c^2 = 225$

$\therefore a^2 + b^2 = c^2$

Exercise 3

Write a formula for $\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$.

Exercise 3 Solution

From our definitions, we know that $\sin\theta = \frac{O}{H}$, $\cos\theta = \frac{A}{H}$ and $\tan\theta = \frac{O}{A}$.

We want to use the fractions $\frac{O}{H}$ and $\frac{A}{H}$ to "create" the fraction $\frac{O}{A}$, which would mean cancelling out the H in the denominators. Using knowledge of fraction division we can try the following:

$$\frac{O}{H} \div \frac{A}{H} = \frac{O}{H} \times \frac{H}{A} = \frac{O}{A} = \tan\theta$$

So we can write tan as a ratio of sin over cos, or: $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

Exercise 4

Try calculating tan(90°) and tan(270°). What did you get? Why does this make sense?

Exercise 4 Solution

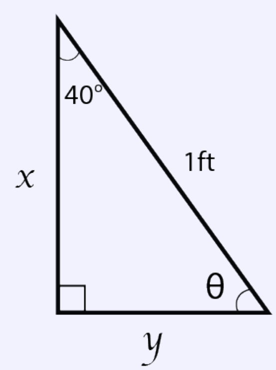
Plugging in tan(90°) and tan(270°) into a calculator gives an error message (depending on your calculator, it will say "Error" or "Math Error"). This indeed makes sense because of our unit circle: at $\theta = 90°$, $\sin\theta = 1$ and $cos\theta = 0$. Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\tan(90°) = \frac{1}{0}$. But this fraction is *undefined* because you can't divide by 0!

Similarly, at $\theta = 270°$, $\sin\theta = -1$ and $cos\theta = 0$. Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\tan(270°) = \frac{-1}{0}$. But this fraction is also undefined due to division by 0.

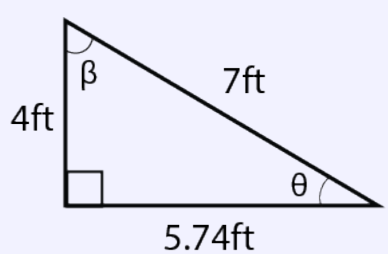
Exercise 5

Solve for all the missing sides and angles of the triangles below:

1. Triangle 🦇



2. Triangle 🕊



Exercise 5 Solution

1. Since we are given two of three angles, we can solve for θ using the sum of angles in a triangle:

$$\begin{aligned} 180^\circ &= 40^\circ + 90^\circ + \theta \\ 180^\circ - 40^\circ - 90^\circ &= \theta \\ 50^\circ &= \theta \end{aligned}$$

   We can now use our trigonometric ratios to solve for x and y:

$$\begin{aligned} \sin(50^\circ) &= \frac{x}{1} \\ \sin(50^\circ) &= x \\ 0.7660 &\approx x \end{aligned}$$

$$\begin{aligned} \cos(50^\circ) &= \frac{y}{1} \\ \cos(50^\circ) &= y \\ 0.6428 &\approx y \end{aligned}$$

2. In this triangle we are given all of the sides and need to determine two angles, so we need to use the inverse of our trigonometric ratios:

$$\begin{aligned} \theta &= \sin^{-1}(\frac{4}{7}) \\ \theta &\approx 34.85^\circ \end{aligned}$$

$$\begin{aligned} \beta &= \cos^{-1}(\frac{4}{7}) \\ \beta &\approx 55.15^\circ \end{aligned}$$

Exercise 6

Prove the Pythagorean Identity.

Hint: Look carefully at the name of this identity, and use the unit circle

Exercise 6 Solution

As we saw in the lesson, for any angle θ, the point on the Unit Circle where the terminal arm of the angle intersects it is $(\cos\theta, \sin\theta)$. Further, if we draw a vertical line down from the point, it creates a right triangle with the x-axis. Since the hypotenuse = radius = 1, the "height" of the triangle is $\sin\theta$ and the "width" of the triangle is $\cos\theta$.

We can apply the Pythagorean Theorem to the sides of the right triangle we created:

$$a^2 + b^2 = c^2$$
$$(\sin\theta)^2 + (\cos\theta)^2 = 1^2$$
$$\sin^2\theta + \cos^2\theta = 1$$

Thus we have proven the Pythagorean Identity.

Exercise 7

Prove the following:

a) $\sec\theta - \tan\theta\sin\theta = \frac{1}{\sec\theta}$

b) $\frac{1+\cos\theta}{\sin\theta} = \csc\theta + \cot\theta$

Exercise 7 Solution

a)

LS	RS
$\sec\theta - \tan\theta\sin\theta$	$\frac{1}{\sec\theta}$
$= \frac{1}{\cos\theta} - \tan\theta\sin\theta$	$= \cos\theta$
$= \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\sin\theta$	
$= \frac{1-\sin^2\theta}{\cos\theta}$	
$= \frac{\cos^2\theta}{\cos\theta}$	
$= \cos\theta$	

$\therefore$ LS = RS, and so we have proven that $\sec\theta - \tan\theta\sin\theta = \frac{1}{\sec\theta}$.

b)

LS	RS
$\frac{1+\cos\theta}{\sin\theta}$	$\csc\theta + \cot\theta$
$= \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}$	
$= \csc\theta + \frac{\cos\theta}{\sin\theta}$	
$= \csc\theta + \cot\theta$	

$\therefore$ LS = RS, and so we have proven that $\frac{1+\cos\theta}{\sin\theta} = \csc\theta + \cot\theta$.

Problem Set Solutions

1. Determine the value of the following trigonometric ratios. Round to four decimal places.

   (a) $\sin\theta, \cos\theta$ if $\theta = 180°$

   (b) $\tan\theta, \cos\theta$ if $\theta = 225°$

   (c) $\sin\theta, \csc\theta$ if $\theta = 15°$

   (d) $\cot\theta, \sec\theta$ if $\theta = 98°$

> *Solution*:
>
> (a) $\sin(180°) = 0$
>
> $\cos(180°) = -1$
>
> (b) $\tan(225°) = 1$
>
> $\cos(225°) \approx -0.7071$
>
> (c) $\sin(15°) \approx 0.2588$
>
> $\csc(15°) \approx 3.8637$

2. Determine θ given the following trigonometric ratios using a calculator. Round to two decimal places.

   (a) $\sin\theta = 0.9659$

   (b) $\tan\theta = -3.7321$

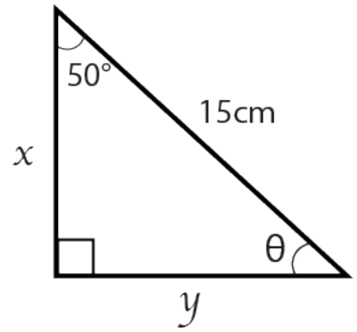
   (c) $\sec\theta = -1.0642$

   (d) $\cos\theta = 1$

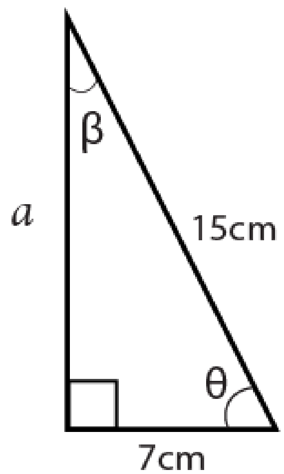
> *Solution*:
>
> (a) $\theta \approx 74.99°$
>
> (b) $\theta = -75°$, OR $105°$, OR $285°$
>
> (c) $\theta = 160°$
>
> (d) $\theta = 0°$, OR $360°$.

3. Solving for any missing angles and/or side lengths:

   (a) Triangle 1



   (b) Triangle 2



Solution:

(a) Find θ using interior angles of a triangle rules:

$$\theta + 50^\circ + 90^\circ = 180^\circ$$
$$\theta = 180^\circ - 90^\circ - 50^\circ$$
$$\theta = 40^\circ$$

Since there are two sides missing, find x and y using trigonometric ratios:

$$\begin{aligned} \sin(40^\circ) &= \frac{x}{15} \\ 15 \times \sin(40^\circ) &= x \\ 9.64 &\approx x \end{aligned} \qquad \begin{aligned} \cos(40^\circ) &= \frac{y}{15} \\ 15 \times \cos(40^\circ) &= y \\ 11.49 &\approx y \end{aligned}$$

$\therefore \quad \theta = 40^\circ, x \approx 9.64\text{cm}, \text{and } y \approx 11.49\text{cm}.$

(b) Since two sides are given, solve for side a using the Pythagorean Theorem:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + 7^2 &= 15^2 \\ a^2 + 49 &= 225 \\ a^2 &= 225 - 49 \\ a^2 &= 176 \\ a &= \sqrt{176} \\ a &\approx 13.27 \end{aligned}$$

Since there are two angles missing, solve for θ and β using inverse trigonometric ratios:

$$\begin{aligned} \sin(\theta) &= \frac{\sqrt{176}}{15} \\ \theta &= \sin^{-1}(\frac{\sqrt{176}}{15}) \\ \theta &\approx 62.18^\circ \end{aligned} \qquad \begin{aligned} \sin(\beta) &= \frac{\sqrt{7}}{15} \\ \beta &= \sin^{-1}(\frac{\sqrt{7}}{15}) \\ \beta &\approx 27.82^\circ \end{aligned}$$

$\therefore \quad \theta \approx 62.18^\circ, \beta \approx 27.82^\circ, \text{and } a \approx 13.27\text{cm}.$

4. A few math students want to get to the top of MC, which is 20 meters tall.

(a) If they have a ladder that is 30 meters long how far do they need to place it from the base of the building? Round your answer to two decimal places.

(b) Draw a diagram representing this problem. Given your answer in part (a), determine the measure of the two missing angles in your diagram. Round your answer to the nearest hundredth of a degree.

Solution:

(a) Using the Pythagorean Theorem:

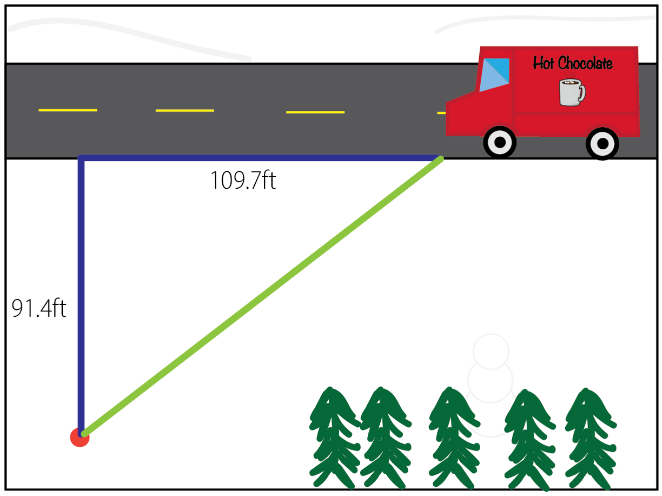
$$\begin{aligned} a^2 + b^2 &= c^2 \\ 20^2 + b^2 &= 30^2 \\ b^2 &= 900 - 400 \\ b^2 &= 500 \\ b &= \sqrt{500} \\ b &\approx 22.36 \end{aligned}$$

(b) The missing angles will be referred to as θ anf β. Using inverse trigonometric ratios:

$$\begin{aligned} \sin(\theta) &= \frac{20}{30} \\ \theta &= \sin^{-1}(\frac{20}{30}) \\ \theta &\approx 41.81^\circ \end{aligned}$$

$$\begin{aligned} \cos(\beta) &= \frac{20}{30} \\ \beta &= \cos^{-1}(\frac{20}{30}) \\ \beta &\approx 138.19^\circ \end{aligned}$$

5. Two friends, Charlie and Daniel, are playing in the snow. They decide to go and get hot chocolate, but take different paths to get to the Hot Chocolate Truck. Charlie thinks it will be faster if he goes to the road in a straight line and then walks along the road to the truck. Daniel thinks it will be faster if he cuts diagonally through the snow.



(a) Looking only at the distance, who will get to the Hot Chocolate Truck faster?

(b) If the speed of walking through the snow is 1.5ft/second and the speed of walking on the road is 5ft/second, does your answer to part (a) change?

Solution:

(a) Charlie's distance, C, is the sum of his distance through the snow and his distance on the road. Daniel's distance, D, is only through the snow, but it is not given, and thus we use Pythagorean Theorem since his way is the hypotenuse of a right triangle.

Solving for C:

$C = 91.4\text{ft} + 109.7\text{ft} = 201.1\text{ft}$

Solving for D:

$$\begin{aligned} 91.4^2 + 109.7^2 &= D^2 \\ 8353.96 + 12034.09 &= D^2 \\ 20388.05 &= D^2 \\ \sqrt{20388.05} &= D \\ 142.79 &\approx D \end{aligned}$$

Since 142.79ft is less than 201.1ft, Daniel will get to the Hot Chocolate Truck faster.

(b) Recall that $time = \frac{distance}{speed}$. Then:

$$\begin{aligned} \text{Charlie's total time} &= \text{Charlie's time through the snow} + \text{Charlie's time on the road} \\ &= \frac{\text{distance through snow}}{\text{speed through snow}} + \frac{\text{distance on road}}{\text{speed on road}} \\ &= \frac{91.4ft}{1.5ft/s} + \frac{109.7ft}{5ft/s} \\ &\approx 60.933s + 21.84s \\ &\approx 82.87 \text{ seconds} \end{aligned}$$

$$\begin{aligned} \text{Daniel's total time} &= \frac{\text{Daniel's distance}}{\text{Daniel's speed}} \\ &= \frac{142.9ft}{1.5ft/s} \\ &\approx 95.19 \text{ seconds.} \end{aligned}$$

$\therefore$ our answer from part a) does change, and Charlie will get to the Hot Chocolate Truck faster.

6. In this word problem we will be looking at **angles of elevation** and **angles of depression**, which are angles between a person's line of sight and the horizontal line (from where they are). It works the same if you have an object instead of a person.

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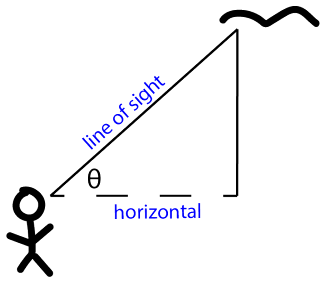
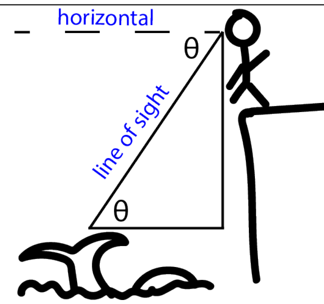
<tr><th>Angle of Elevation</th><th>Angle of Depression</th></tr>
<tr><td>Angle measured above the horizontal</td><td>Angle measured below the horizontal</td></tr>
<tr><td>
</td><td>
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A cliff is 72 m high, and a boat travelling towards the cliff is 80 m from the base of the cliff.

(a) Determine the angle of depression from the cliff to the boat, to the nearest tenth of a degree.

(b) If the angle of depression from the cliff to the boat had to be 50°, how much closer would the boat have to move towards the cliff? Round your answer to the nearest tenth of a meter.

Solution:

(a) The question provides the "opposite" and "adjacent" values, so the inverse tan trigonometric function can be used:

$$\tan\theta = \frac{72}{80}$$
$$\theta = \tan^{-1}(\frac{72}{80}$$
$$\theta \approx 42.0^\circ$$

$\therefore$ the angle of depression from the cliff to the boat is approximately 42.0°.

(b) First, determine what the distance is from the cliff to the boat if the angle of depression is 50°:

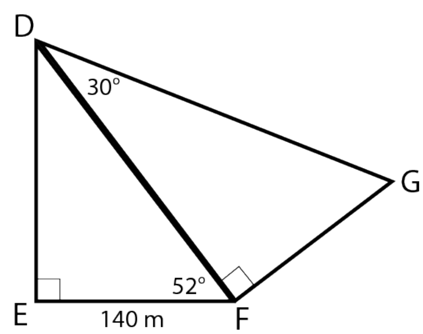
$$\tan(50^\circ) = \frac{72}{x}$$
$$x = \frac{72}{\tan 50^\circ}$$
$$x \approx 60.4m$$

Now, calculating the difference between the two distances: 80 - 60.4 $\approx$ 19.6m.

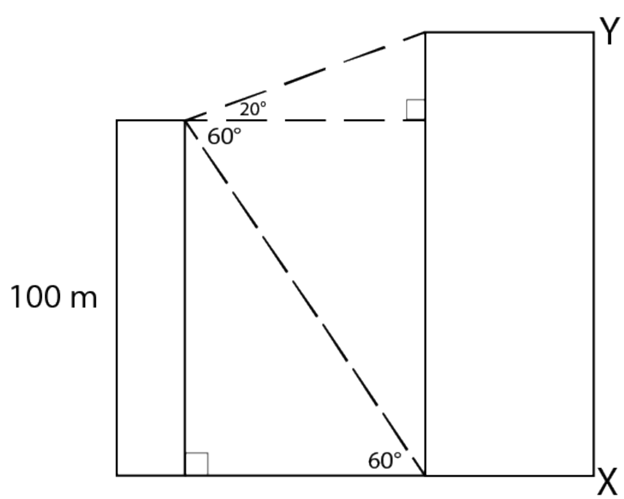
$\therefore$ the boat would have to be approximately 19.6m closer to the cliff if the angle of depression had to be 50°.

7. Find the length of the following to the nearest tenth of a meter:

   (a) Side DG of this shape:



   (b) Side XY of this shape:



Solution:

(a) To find DG we need at least one of DF or GF. Fortunately, side DF can be found using triangle DEF:

$$\cos 52^\circ = \frac{140}{DF}$$
$$DF = \frac{140}{\cos 52^\circ}$$
$$DF \approx 227.40m$$

Now looking at triangle DFG we see that:

$$\cos 30^\circ = \frac{227.40}{DG}$$
$$DG = \frac{227.40}{\cos 30^\circ}$$
$$DG \approx 262.6m$$

(b) Side XY of this shape is 100m + a smaller side, which we will call a. Before we find a, we must find the width of the middle rectangle, which we will call b:

$$\tan 60^\circ = \frac{100}{y}$$
$$y = \frac{100}{\tan 60^\circ}$$
$$y \approx 57.74m$$

We use the tangent trigonometric function once more to find x:

$$\tan 20^\circ = \frac{x}{57.7}$$
$$x = 57.7\tan(20^\circ)$$
$$x \approx 21.00m$$

Going back to what we noted in the beginning, XY = 100 + x = 100 + 21 = 121m.

8. Prove the following:

(a) $\sin\theta = \cos\theta\tan\theta$

(b) $\sin\theta\tan\theta + \cos\theta = \frac{1}{\cos\theta}$

(c) $\frac{1}{\cos^2\theta} = \tan^2\theta + 1$

Solution:

a)

LS	RS
$\sin\theta$	$\cos\theta\tan\theta$
	$= \cos\theta \times \frac{\sin\theta}{\cos\theta}$
	$= \sin\theta$

$\therefore$ LS = RS, and so we have proven that $\sin\theta = \cos\theta\tan\theta$.

b)

LS	RS
$\sin\theta\tan\theta + \cos\theta$	$\frac{1}{\cos\theta}$
$= \sin\theta \times \frac{\sin\theta}{\cos\theta} + \cos\theta$	
$= \frac{\sin^2\theta}{\cos\theta} + \frac{\cos^2\theta}{\cos\theta}$	
$= \frac{\sin^2\theta+\cos^2\theta}{\cos\theta}$	
$= \frac{1}{\cos\theta}$	

$\therefore$ LS = RS, and so we have proven that $\sin\theta\tan\theta + \cos\theta = \frac{1}{\cos\theta}$.

c)

LS	RS
$\frac{1}{\cos^2\theta}$	$\tan^2\theta + 1$
	$= \frac{\sin^2\theta}{\cos^2\theta} + 1$
	$= \frac{\sin^2\theta+\cos^2\theta}{\cos^2\theta}$
	$= \frac{1}{\cos^2\theta}$

$\therefore$ LS = RS, and so we have proven that $\frac{1}{\cos^2\theta} = \tan^2\theta + 1$.